Exercise 1.5.13

Determine the *steady-state* temperature distribution between two concentric *spheres* with radii 1 and 4, respectively, if the temperature of the outer sphere is maintained at 80° and the inner sphere at 0° (see Exercise 1.5.12).

Solution

The governing equation for the temperature between the two spheres, assuming radial symmetry, is

$$\frac{\partial u}{\partial t} = \frac{k}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right), \quad 1 \le r \le 4.$$

At equilibrium the temperature does not change in time, so $\partial u/\partial t$ vanishes. u is only a function of r now.

$$0 = \frac{k}{r^2}\frac{d}{dr}\left(r^2\frac{du}{dr}\right) \quad \to \quad \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{du}{dr}\right) = 0$$

To solve the differential equation, multiply both sides by r^2.

$$\frac{d}{dr}\left(r^2\frac{du}{dr}\right) = 0$$

Integrate both sides with respect to r.

$$r^2\frac{du}{dr} = C_1$$

Divide both sides by r^2.

$$\frac{du}{dr} = \frac{C_1}{r^2}$$

Integrate both sides with respect to r once more.

$$u(r) = -\frac{C_1}{r} + C_2$$

Now we apply the two boundary conditions to determine C_1 and C_2:

$$\begin{aligned} u(r=1) &= -C_1 + C_2 = 0 \\ u(r=4) &= -\frac{C_1}{4} + C_2 = 80. \end{aligned}$$

Solving this system of equations yields $C_1 = 320/3$ and $C_2 = 320/3$. Therefore, the steady-state temperature distribution is

$$u(r) = \frac{320}{3}\left(1 - \frac{1}{r}\right).$$